The optimal spending rate versus the expected real return of a sovereign wealth fund.

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Lecture at:
Conference in Memory of Tomas Bjørk.
Stockholm, Sverige - October 10 - 11, 2022.

Abstract

- We consider a sovereign wealth fund that invests broadly in the international financial markets. We adopt the life cycle model and demonstrate that the optimal spending rate from the fund is significantly less than the fund's expected real rate of return. The optimal spending rate secures that the fund will last "forever" (under certain conditions).

- Spending the expected return will deplete the fund with probability one. Moreover, this strategy is inconsistent with optimal portfolio choice. Our results are contrary to the idea that it is sustainable to spend the expected return of a sovereign wealth fund.

- As an example, we consider the Norwegian Oil Fund, or formally the Norwegian SWF Government Fund Global.

Introduction

- We consider optimal investment strategies and the associated optimal spending from an endowment fund consistent with the life cycle model. We demonstrate that the optimal spending rate is strictly smaller than the expected rate of return. The difference is far from negligible, and amounts to several percentage points in most real situations.

- One basic explanation: If the fund is managed by diversification, this means that risk aversion plays an essential part in the optimal portfolio choice problem. Then, to be consistent, the spending rate must also reflect this. As a consequence, the expected real rate of return is typically not an optimal spending rate, since this criterion would normally be associated with risk neutrality.

The Basic Model

- We consider the optimal consumption and portfolio selection problem using the life cycle model. We have an agent represented by the pair (U, e), where $U(c)$ is the agent's utility function over consumption processes c, and e is the agent's endowment process. The problem consists in maximizing utility subject to the agent's budget constraint
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$$\sup_{c,\varphi} U(c) \quad \text{subject to} \quad E\Big(\int_0^T \pi_t c_t dt\Big) \leq E\Big(\int_0^T \pi_t e_t dt\Big) := w, \tag{1}$$
- Here φ are the optimal fractions of wealth in the various investment possibilities facing the agent, and w is the current value of the agent's wealth. The quantity π_t is the state price deflator at each time t, i.e., the discounted Arrow-Debreu state prices in units of probability. The horizon is $T \leq \infty$.

- The consumer takes as given a dynamic financial market, consisting of N risky securities and one riskless asset, the latter with rate of return r_t, a stochastic process. The agent's actions do not affect market prices of the risky assets, nor the risk-free rate of return r_t.

- There are no arbitrage possibilities in the financial market (market price of risk is bounded, there is an equivalent martingale measure Q, no arbitrage in a suitable space of strategies).

Optimal consumption and spending with expected utility

- We consider a continuous-time framework. In case (i) the agent's preferences are represented by standard expected additive and separable utility of the form

$$U(c) = E\Big(\int_0^T u(c_t, t) dt \Big). \tag{2}$$

- Here $u(c,t)$ is the agent's felicity index, which we assume to be of the CRRA-type: $u(x,t) = \frac{1}{1-\gamma} x^{1-\gamma} e^{-\delta t}$, where γ is the agent's relative risk aversion and δ is the agent's impatience rate (the utility discount rate).

- The financial market consists of N risky assets, where

$$\sigma\eta = \lambda = (\mu_1 - r, \mu_2 - r, \ldots, \mu_N - r)'$$

- Here the matrix $\sigma_t \sigma_t'$ is the instantaneous variance/covariance matrix of the risky assets in units of prices.

- $\eta =$ market price of risk, and $\lambda =$ excess returns of the assets.

- The connection $\eta_t' \eta_t = \lambda_t' (\sigma_t \sigma_t')^{-1} \lambda_t$ follows.

- The dynamics of the wealth portfolio W_t. It is given by the following stochastic differential equation

$$dW_t = [W_t(\varphi_t' \lambda_t + r_t) - c_t]dt + W_t \varphi_t' \sigma_t dB_t, \quad W_0 = w. \qquad (3)$$

- It follows from optimal consumption and portfolio choice theory that the optimal consumption per time unit, c_t^*, and the optimal wealth at time t, W_t^*, are connected.

- The starting point for this derivation is the following formula for the market value of current wealth W_t

$$W_t^* = \frac{1}{\pi_t} E_t \Big\{ \int_t^T \pi_s c_s^* ds \Big\}. \tag{4}$$

(Recall π_t is the state price deflator.)

- Under the assumption of no arbitrage possibilities, π_t is given by

$$\pi_t = e^{-\int_0^t (r_u + \frac{1}{2}\eta_u' \eta_u) du - \int_0^t \eta_u dB_u} \tag{5}$$

- B_t is a standard d-dimensional Brownian motion, which generates the information set $\mathcal{F}_t$ for all $t \in [0, T]$. We assume $d = N$.

A deterministic investment opportunity set. Some basic results

- By employing Kuhn-Tucker and the Saddle Point Theorem, we find that the optimal consumption is given by

$$c_t^* = \pi_t^{-\frac{1}{\gamma}} (\alpha e^{\delta t})^{-\frac{1}{\gamma}}, \tag{6}$$

  where α is the Lagrange multiplier, ultimately determined by equality in the budget constraint.

- With a deterministic investment opportunity set, the optimal portfolio weights at any time t are given by

$$\varphi_t = \frac{1}{\gamma} (\sigma \sigma')^{-1} \lambda \quad \text{for all } t. \tag{7}$$

- Let $\mathcal{I}_t = (r_t, \eta_t, \lambda_t)$ signify the investment opportunity set.

- We can write the optimal wealth in equation (4) of the agent in terms of the optimal consumption as follows

$$W_t^* = c_t^* E_t \Big\{ \int_t^T e^{\frac{1-\gamma}{\gamma}[\int_t^s ((r_u + \frac{1}{2}\eta_u' \eta_u) - \frac{\delta}{1-\gamma}) du + \int_t^s \eta_u dB_u]} ds \Big\}, \quad (8)$$

- We have used the dynamics for the state price deflator in (5) and that for the optimal consumption which follows from (6).

- The optimal consumption to wealth ratio, $\frac{c_t^*}{W_t^*} = k_t$, is the optimal spending rate.

- We then have the following result

- **Proposition 1** *Assuming a deterministic investment opportunity set, the optimal spending rate k is a constant and depends on the return from the fund only via the "certainty equivalent" rate of return, and can be written*

$$k = \frac{\delta}{\gamma} + \left(1 - \frac{1}{\gamma}\right)\left(r + \frac{1}{2}\gamma\varphi'(\sigma\sigma')\varphi\right). \tag{9}$$

- The optimal spending rate is seen to be a convex combination between the impatience rate δ and the quantity $\left(r + \frac{1}{2}\gamma\varphi'(\sigma\sigma')\varphi\right)$.

- The latter is equal to the *certainty equivalent rate of return.*

- It is really the Arrow-Pratt approximation to this quantity.

- In continuous-time models with Brownian-driven uncertainty, this type of approximation becomes exact.

- One basic comparison is between the expected real rate of return on the wealth portfolio, which is

$$E_t\big(dR_t\big)/dt = r + \frac{1}{\gamma}\lambda'(\sigma\sigma')^{-1}\lambda. \tag{10}$$

  and the optimal extraction rate k.

- Assuming an infinite horizon for now, the inequality

$$k \leq r + \frac{1}{\gamma}\lambda'(\sigma\sigma')^{-1}\lambda \tag{11}$$

  holds if and only if

$$r \geq \delta - \lambda'\varphi\Big(\frac{1+\gamma}{2\gamma}\Big). \tag{12}$$

- Since the second term on the right-hand side is negative, this inequality is true for reasonable values of the parameters of this problem.

- John Campbell (2012) recommends that the spending rate is set equal to the real expected rate of return. In the author's own words.

- *"The sustainable spending rate of an endowment, which is the amount spent as a fraction of the market value of the endowment, must equal the expected return in order to achieve immortality."*

- This is called "vigorous immortality" by the author.

- As we have just demonstrated, this policy is a little bit too vigorous to be rational and consistent, and implies a contradiction. This policy will eventually deplete the fund with probability 1, shown in the paper.

- Dybvig and Qin (2019) consider a fund with normal iid log-returns. The authors find that for the fund to last "forever", spending must not exceed expected fund return subtracted by half the variance. The discrepancy between expected fund return and sustainable spending is far from negligible.

- Can the policy advocated by Dybvig and Quin (2019), also considered in Campbell and Sigalov (2020), be consistent with the optimal spending rule outlined in the above?

- A little analysis shows that this requires $r = \delta$ and $\gamma = 0$, but the latter is not allowed in our model. Accordingly is the criterion of expected fund return subtracted by half the variance not optimal for valid values of the preference parameters.

- Campbell and Martin (2022) seem to recognise that the consumption to wealth ratio is the correct quantity to focus on, and introduces a sustainablility constraint. No reference to our paper.

- Consider the data of more than 100 years related to the $S\&P$-500 index, used by Mehra and Prescott (1985) in their well-known study.

- For $\gamma = 2.5$ and $\delta = 0.01$, it follows that $\varphi = 0.95$, the expected rate of return on the wealth portfolio is 0.065, the certainty equivalent rate of return is 0.037 which gives an optimal spending rate of 0.026.

The asymptotic behavior of a sovereign wealth fund

- When the spending rate k is a constant (not necessarily optimal), as in the above model, the wealth W_t is a geometric Brownian motion with dynamics

$$W_t = W_0 e^{\int_0^t [\mu_W - \frac{1}{2}\varphi'(\sigma\sigma')\varphi] ds + \int_0^t \varphi' \sigma dB_s}, \tag{13}$$

where

$$\mu_W = \begin{cases} 0, & \text{if } k = r + \gamma\varphi'(\sigma\sigma')\varphi; \\ \frac{1}{2}(1+\gamma)\varphi'(\sigma\sigma')\varphi + \frac{1}{\gamma}(r-\delta), & \text{if } k \text{ is optimal.} \end{cases} \tag{14}$$

- This can be used to both study convergence in 1. mean and convergence a.s. In the first case we have:

- If $\mu_W > 0$ the process W_t is a submartingale, in which case $E_t(W_s) \geq W_t$ for all $s \geq t$; if $\mu_W < 0$ the process W_t is a supermartingale, in which case $E_t(W_s) \leq W_t$ for all $s \geq t$.

- We have the former, $\mu_W > 0$, if $\delta < \frac{1}{2}(1+\gamma)\gamma\varphi'(\sigma\sigma')\varphi + r$, and the latter, $\mu_W < 0$, if $\delta > \frac{1}{2}(1+\gamma)\gamma\varphi'(\sigma\sigma')\varphi + r$.

- When it comes to convergence a.s., we can summarize as follows:

- **Theorem 1** *(i) With the optimal spending rate k, the fund value W_t goes to infinity (a.s.) as $t \to \infty$ as long as the impatience rate δ is smaller than or equal to the certainty equivalent rate of return on the fund, assuming $\gamma \geq 1$.*

  (ii) If the spending rate is set equal to the expected rate of the return on the fund, then the fund value goes to 0 with probability 1 as time goes to infinity.

- The martingale property gives that $E(W_t) = W_0$ for all $t \geq 0$, but despite of this the wealth eventually converges to zero with probability 1, by the above result, when the spending rate is that of the expected rate of return on the fund.

- Thus, when $\mu_W = 0$, i.e., when spending equals the expected return as advocated by e.g., Campbell (2012), the fund value converges to zero almost surely.

- Based on the above, we find this "recommendation" a bit embarrassing.

The Norwegian Government Pension Fund Global

- As an example consider again the Norwegian Government Pension Fund Global, formerly simply the Norwegian Oil Fund, from the above perspective.

- At the end of 2021 the market value of this fund was 1299 billions USD.

- The idea of the origins of this fund is that also future generations are supposed to benefit from the oil exploration of the present generation, not only those who live in Norway at the present.

- Despite of the change of the name of the former Norwegian Oil Fund, the actual daily use of this fund seems to be more in line with the description considered in this paper.

- The conclusion is to use the separation principle and treat this fund in isolation, where an optimal extraction policy must be consistent with the portfolio selection strategy used.

- Since this is one of broadly diversifying over assets in international security markets, including various government bonds, and also real estate, it is clear that this implies risk aversion on the investment side.

- The preference for diversification is intrinsically equivalent to risk aversion (Louis Eckhoudt, Christian Gollier, Harris Schlesinger (2005)).

- Consistent with this, the optimal extraction rate should also take into account both risk aversion, consumption substitution and impatience, as explained in the paper.

- This is contrary to the current state of affairs of the Norwegian Government Pension Fund Global, where the extraction from this fund is determined by a mandate from the Parliament (Stortinget) to be *set equal to the expected real return on the fund.*

- As we have shown, this is not the sustainable, or viable spending rate of a fund like this, and will deplete the fund in the future with probability one.

Some expectations for the Norwegian SWF Government Fund Global

- For this sovereign fund the Norwegian Ministry of Finance set down a commission in 2016, led by Knut Anton Mork, to consider the asset allocation problem. Table 1 below reflects the commission's market view on equity and risky bonds.

	Expectation	Standard dev.	Covariance
Equity	3.4%	16.00%	0.00384
Bonds	0.49%	6.00%	
Equity premium	2.90%	14.67%	

Table 1: The commission's market view, Norwegian Ministry of Finance (2016).

- The commission recommends an equity share of 70% ($\varphi = 0.70$).

- Given a riskless rate of 0.49% and an equity premium with expectation 2.9% and standard deviation 16%, this translates into an implicit risk aversion of $\gamma = 1.61$.

- The expected return and standard deviation of the fund are then 2.53% and 11.25%, respectively.

- The certainty equivalent fund return is $ce_\gamma = 1.79\%$.

- Observe that the certainty equivalent fund return is substantially less than the current fiscal rule, which is 3%.

- Suppose $\delta = 0.01$. Then the optimal long term spending rate with *expected utility* is 0.012, which passes both the long run tests. This is 1.4% lower than the expected real return on the fund.

Recursive utility

- We have also studied recursive utility.

Proposition 2 *With recursive utility, assuming a deterministic investment opportunity set, the optimal extraction rate k is a constant and depends on the return from the fund only via the certainty equivalent rate of return. It is given by*

$$k = \frac{\delta}{\rho} + \big(1 - \frac{1}{\rho}\big)\big(r + \frac{1}{2}\gamma\varphi'(\sigma\sigma')\varphi\big). \tag{15}$$

- The expected real rate of return on the fund is larger than or equal to the optimal extraction rate if and only if the inequality (16) holds, where

$$\frac{r}{\rho} \geq \frac{\delta}{\rho} - \frac{1+\rho}{2\rho\gamma}\lambda'(\sigma\sigma')^{-1}\lambda. \tag{16}$$

- Since the second term on the right-hand side is negative, this inequality holds true for all reasonable values of the parameters, just as in the case of expected utility.

- One new parameter is seen to occur: $\rho = 1/EIS$. (EIS = the elasticity of intertemporal substitution in consumption.)

- The parameter ρ measures resistance against substitution consumption across time. Large values of ρ translates to higher spending rates.

- Assuming $\rho = 2.0$ and $\delta = 0.01$, $\gamma = 1.61$, then $k = 0.013$ and we have preference for late resolution of uncertainty.

- When $\rho = 1.5$ we have $\gamma > \rho$, preference for early resolution of uncertainty, and the optimal long term spending is $k = 0.012$, assuming the other parameters are as above.

- In Figure 1 we illustrate the optimal long term spending rate as a function of the parameter ρ for the above data.

- The upper line represents the expected rate of return on the fund $\mu_W = 0.026$. The next line is $m = 0.025$ (the martingale threshold), then follows $ce_1 = 0.019$ (the almost sure threshold), and finally $ce_\gamma = 0.016$ (the certainty equivalent rate of return).

- The optimal spending rate k is the lowest increasing curve, and passes both the long run tests.

- In contrast, the expected rate of return on the fund, as a spending rate, does not pass either test.

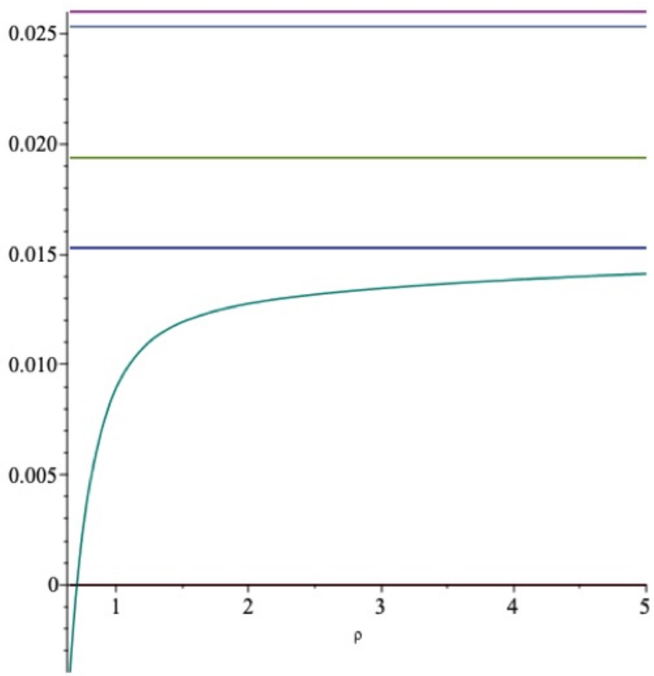

Fig. 1: The optimal spending rate k as a function of ρ.

- We illustrate with a concrete values of ρ. In Figure 2 the parameter $\rho = 1.5$, $\delta = 0.02$ and $\gamma = 1.61$ as above.

- This implies preference for early resolution of uncertainty. Then we have the illustration in Figure 2: The lines are the same as before, (γ and φ are the same).

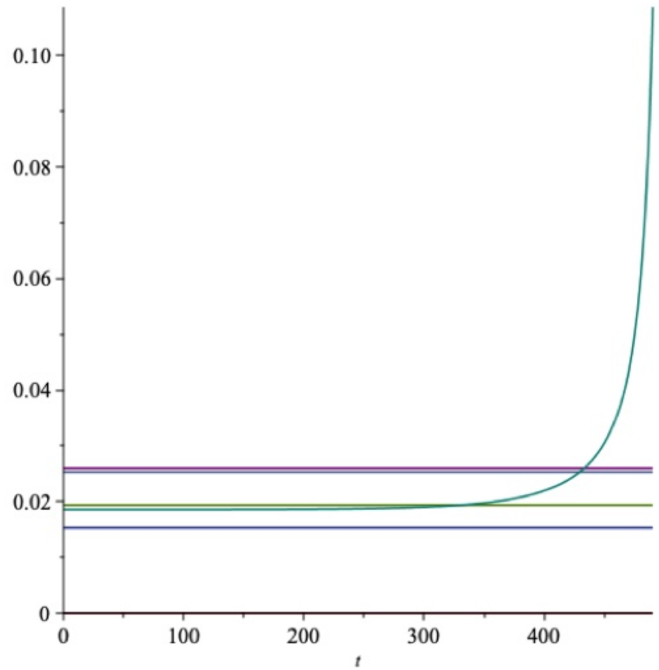

Fig. 2: The optimal extraction rate as a function of time (RU).

- The optimal long run extraction rate is $k < 0.019$ (slightly), for RU. The hyperbolic type curve is the extraction rate $c_t(500), 0 \leq t < 500$.

- This rate passes all the tests up almost to the end. The optimal long run spending rate with EU is here $k = 0.018$.

- Since δ has here increased, the optimal spending rate is now "living dangerously close" to the a.s. threshold. (Discrete time result.)

- Our paper is published in *Journal of Risk and Financial Management*, 2021, Volume 14, Issue 9, pp 2-35, 425//doi.org/10.3390/jrfm14090425.

- ”Handlingsregelen må baseres på et optimalt uttak fra Oljefondet.” (in Norwegian). With Petter Bjerksund. Dagens Næringsliv (DN). Publshed on net 14.09.21-18.08. In paper: 15.09.21. 2021. An explanation for the public of the main results in this publication.

- The latter part has some elements from a discrete time, discrete space version: ”Optimal spending of a wealth fund in the discrete time life cycle model.” K.K. Aase (2022).

- In the latter version a quantity $\tilde{ce}_\gamma$ is defined as

$$\tilde{ce}_\gamma := E(R) - \frac{1}{2}\gamma E(R^2).$$

where R be the simple return.

- As long as $\gamma > 1$, $\tilde{ce}_\gamma < ce_1 := E(R) - \frac{1}{2}E(R^2)$ which is the crucial test to avoid almost sure convergence of the fund value towards 0 as t increases. Obviously $\tilde{ce}_\gamma < E(R)$.

- In general $\tilde{ce}_\gamma$ is what we might suggest, for reasons of simplicity, as the responsible spending rate. The reason is that the only parameter required, except from market data, is the relative risk aversion γ.

- This parameter is implicitly given once the fraction φ is determined. This quantity is suggested, more or less, in the Norwegian parliament.

Fig. 3: Outside Universidad Ciudad, Mexico City, 1998.

- We can see Tomas in the middle, Ole Barndorff Nielsen to the left, Bernt

Øksendal, Thomas, Knut Aase and Paul Embrechts to the far right.